

RIEMANNIAN SUBMERSIONS WITH DISCRETE SPECTRUM

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ABSTRACT. We prove some estimates on the spectrum of the Laplacian of the total space of a Riemannian submersion in terms of the spectrum of the Laplacian of the base and the geometry of the fibers. When the fibers of the submersions are compact and minimal, we prove that the total space is discrete if and only if the base is discrete. When the fibers are not minimal, we prove a discreteness criterion for the total space in terms of the relative growth of the mean curvature of the fibers and the mean curvature of the geodesic spheres in the base. We discuss in particular the case of warped products.

1. INTRODUCTION

Let M be a complete Riemannian manifold and $\Delta = \operatorname{div} \circ \operatorname{grad}$ be the Laplace-Beltrami operator acting on the space of smooth functions on M with compact support. The operator Δ is essentially self-adjoint, thus it has a unique self-adjoint extension, to an unbounded operator, denoted by Δ , whose domain is the set of functions $f \in L^2(M)$ so that $\Delta f \in L^2(M)$. Recall that the spectrum of a self-adjoint operator A , denoted by $\sigma(A)$, is formed by all $\lambda \in \mathbb{R}$ for which $A - \lambda I$ is not injective or the inverse operator $(A - \lambda I)^{-1}$ is unbounded, [7]. In this paper we are going to study the spectrum of $-\Delta$, (the operator Δ is negative), and we refer to $\sigma(-\Delta)$ as the spectrum of M and in this case only, we denote by $\sigma(M)$. It is important (in our study) to distinguish the various types of elements of the spectrum of M in order to have a better understanding of the relations between M and $\sigma(M)$. This way, it is said that the set of all eigenvalues of $\sigma(M)$ is the *point spectrum* $\sigma_p(M)$, while the *discrete spectrum* $\sigma_d(M)$ is the set of all isolated¹ eigenvalues of finite multiplicity. The *essential spectrum* $\sigma_{\text{ess}}(M) = \sigma(M) \setminus \sigma_d(M)$ is the complement of the discrete spectrum.

There is a vast literature studying the spectrum of complete Riemannian manifolds, among that, we point out geometric restrictions implying that the spectrum is purely continuous ($\sigma_p(M) = \emptyset$), see [9], [10], [12], [16], [21], [23] or implying that the spectrum is discrete ($\sigma_{\text{ess}}(M) = \emptyset$), see [1], [4], [11], [15], [17], [18].

Date: January 5th, 2010.

The authors were partially supported by CNPq-CAPES (Brazil) and MEC project PCI2006-A7-0532 (Spain).

¹Isolated in the sense that for some $\varepsilon > 0$ one has that $(\lambda - \varepsilon, \lambda + \varepsilon) \cap \sigma(\Delta^M) = \lambda$.

In [1], Baider studied the essential spectrum of warped product manifolds $W = X \times_\gamma Y = (X \times Y, dX^2 + \gamma^2(x)dY^2)$, where $\gamma: X \rightarrow \mathbb{R}$ is a positive smooth function. The Laplace-Beltrami operator Δ^W restricted to $C_0^\infty(X) \otimes C_0^\infty(Y)$ has this form $\Delta^W = A_0 \otimes 1_Y + \gamma^{-2} \otimes (\Delta^Y)$, where A_0 is an elliptic operator, symmetric relative to the density $\gamma^n dX$ with the same symbol as Δ^X . Baider showed that if $\sigma_{\text{ess}}(Y) = \emptyset$ then

$$(1.1) \quad \sigma_{\text{ess}}(W) = \emptyset \iff \sigma_{\text{ess}}(A_1) = \emptyset,$$

where $A_1 = A_0 + \lambda^*(Y)\gamma^{-2}$ and $\lambda^*(Y) = \inf \sigma(\Delta^Y)$. When $\gamma \equiv 1$ then $W = X \times Y$ and $A_1 = \Delta^X + \lambda^*(Y)$. In general, one can not substitute $\sigma_{\text{ess}}(A_1) = \emptyset$ by $\sigma_{\text{ess}}(X) = \emptyset$. There are examples of warped manifolds $\mathbb{R}^n \times_\gamma \mathbb{S}^1$, with discrete spectrum, therefore $\sigma_{\text{ess}}(A_1) = \emptyset$, see [1], but the spectrum $\sigma_{\text{ess}}(\mathbb{R}^n) = [0, \infty)$. Riemannian manifolds whose Laplacian has empty essential spectrum are sometimes called *discrete* in the literature.

In this paper we consider Riemannian submersions $\pi: M \rightarrow N$ and we prove some spectral estimates relating the (essential) spectrum of M and N . Riemannian submersions were introduced in the sixties by B. O'Neill and A. Gray (see [13, 19, 20]) as a tool to study the geometry of a Riemannian manifold with an additional structure in terms of certain components, that is, the fibers and the base space. When M (and thus also N) is compact, estimates on the eigenvalues of the Laplacian of M have been studied in [5], under the assumption that the mean curvature vector of the fibers is *basic*, i.e., π -related to some vector field on the basis. We will consider here the non compact case, assuming initially that the fibers are minimal.

An important class of examples are Riemannian homogeneous spaces G/K , where G is a Lie group endowed with a bi-invariant Riemannian metric and K is a closed subgroup of G , see [19] for details. The projection $G \rightarrow G/K$ is a Riemannian submersions with totally geodesic fibers, and with fibers diffeomorphic to K .

Another important class of examples of manifolds that can be described as the total space of Riemannian submersions with minimal fibers are the homogeneous 3-dimensional Riemannian manifolds with isometry group of dimension four, see [22]. This class includes the special linear group $\text{SL}(2, \mathbb{R})$ endowed with a family of left-invariant metrics, which is the total space of Riemannian submersions with base given by the hyperbolic spaces, and fibers diffeomorphic to \mathbb{S}^1 .

Given a Riemannian submersion $\pi: M \rightarrow N$ with compact minimal fibers, we prove that

$$\sigma_{\text{ess}}(M) = \emptyset \iff \sigma_{\text{ess}}(N) = \emptyset,$$

see Theorem 1. This result coincides with Baider's result when $M = X \times Y$ is a product manifold, Y is compact, $N = X$ and $\pi: X \times Y \rightarrow X$ is the projection on the first factor.

Theorem 1. *Let $\pi: M \rightarrow N$ be a Riemannian submersion with compact minimal fibers. Then*

- i. $\sigma_{\text{ess}}(N) \subset \sigma_{\text{ess}}(M)$, $\sigma_p(N) \subset \sigma_p(M)$, *thus* $\sigma(N) \subset \sigma(M)$.
- ii. $\inf \sigma_{\text{ess}}(N) = \inf \sigma_{\text{ess}}(M)$. *Therefore, M is discrete if and only if N is discrete.*

A few remarks on this result are in order. First, we observe that for the inequality $\inf \sigma_{\text{ess}}(M) \leq \inf \sigma_{\text{ess}}(N)$, Lemma 3.7, we need only the compactness of the fibers with uniformly bounded volume, meaning that $0 < c^2 \leq \text{vol}(\mathcal{F}_p) \leq C^2$ for all $p \in N$. Second, the example of [1] shows that the assumption of minimality of the fibers is necessary in Theorem 1. In fact, one has examples of Riemannian submersions having compact fibers with discrete base and non discrete total space, or with discrete total space but not discrete base, see Example 4.2.

In the second part of the paper we study the essential spectrum of the total space when the minimality assumption on the fiber is dropped. In this case, we prove that a sufficient condition for the discreteness of the total space is that the growth of the mean curvature of the fibers at infinity is controlled by the growth of the mean curvature of the geodesic spheres in the base manifold. In order to state our result, let us introduce the following terminology. The cut locus $\text{cut}(p)$ of a point p in a Riemannian n -manifold is said to be *thin*, if its $(n-1)$ -Hausdorff measure zero, $\mathcal{H}^{n-1}(\text{cut}(p)) = 0$.

Theorem 2. *Let $\pi: M \rightarrow N$ be a Riemannian submersion with compact fibers, and assume that N has a point x_0 with thin cut locus. If the function $h: M \rightarrow \mathbb{R}$ defined by*

$$h(q) = (\Delta^N \rho_{p_0})_{\pi(q)} + g^N((\text{grad}^N \rho_{p_0})_{\pi(q)}, d\pi_q(H_q))$$

is proper then $\sigma_{\text{ess}}(M) = \emptyset$. Here ρ_{p_0} is the distance function in N to p_0 .

The Theorem 2 can be interpreted geometrically in terms of the mean curvature of the geodesic spheres in the base and the mean curvature of the fibers. Namely, the Laplacian of the distance function $\rho_{p_0}(p)$ is exactly the value of the mean curvature of the geodesic sphere $\mathcal{S}_p = \rho_{p_0}^{-1}(\rho_{p_0}(p))$ at the point p . Thus, assumption says that the sum of the mean curvature of the geodesic balls in N and the mean curvature of the fibers must diverge at infinity.

Theorem 1 is proved in Section 3 and Theorem 2 in Section 4. An alternative statement of Theorem 2 can be given in terms of radial curvature, see Corollary 4.2. There are two basic ingredients for the proof of our results.

- The Decomposition Principle, that relates the fundamental tone of the complement of compact sets with the infimum of the essential spectrum, see Proposition 3.2;
- Two estimates of the fundamental tones of open sets in terms of the divergence of vector fields, proved recently in [2] and [3], see Propositions 3.4 and 3.5.

2. RIEMANNIAN SUBMERSIONS

2.1. Preliminaries. Given manifolds M and N , a smooth surjective map $\pi: M \rightarrow N$ is a submersion if the differential $d\pi(q)$ has maximal rank for every $q \in M$. If $\pi: M \rightarrow N$ is a submersion, then for all $p \in N$ the inverse image $\mathcal{F}_p = \pi^{-1}(p)$ is a smooth embedded submanifold of M , that will be called the *fiber* at p . If M and N are Riemannian manifolds, then a submersion $\pi: M \rightarrow N$ is called a *Riemannian submersion* if for all $p \in N$ and all $q \in \mathcal{F}_p$, the restriction of $d\pi(q)$ to the orthogonal subspace $T_q\mathcal{F}_p^\perp$ is an isometry onto T_pN .

Given $p \in N$ and $q \in \mathcal{F}_p$, a tangent vector $\xi \in T_qM$ is said to be *vertical* if it is tangent to \mathcal{F}_p , and it is *horizontal* if it belongs to the orthogonal space $(T_q\mathcal{F}_p)^\perp$. Let $\mathcal{D} = (T\mathcal{F})^\perp \subset TM$ denote the smooth rank k distribution on M consisting of horizontal vectors. The orthogonal distribution \mathcal{D}^\perp is clearly integrable, the fibers of the submersion being its maximal integral leaves. Given $\xi \in TM$, its horizontal and vertical components are denoted respectively by ξ^h and ξ^v . The second fundamental form of the fibers is a symmetric tensor $\mathcal{S}^\mathcal{F}: \mathcal{D}^\perp \times \mathcal{D}^\perp \rightarrow \mathcal{D}$, defined by

$$\mathcal{S}^\mathcal{F}(v, w) = (\nabla_v^M W)^h,$$

where W is a vertical extension of w and ∇^M is the Levi-Civita connection of M .

For any given vector field $X \in \mathfrak{X}(N)$, there exists a unique horizontal $\tilde{X} \in \mathfrak{X}(M)$ which is π -related to X , this is, for any $p \in N$ and $q \in \mathcal{F}_p$, then $d\pi_q(\tilde{X}_q) = X_p$, called *horizontal lifting* of X . A horizontal vector field $\tilde{X} \in \mathfrak{X}(M)$ is called *basic* if it is π -related to some vector field $X \in \mathfrak{X}(N)$.

If \tilde{X} and \tilde{Y} are basic vector fields, then these observations follows easily.

- (a) $g^M(\tilde{X}, \tilde{Y}) = g^N(X, Y) \circ \pi$.
- (b) $[\tilde{X}, \tilde{Y}]^h$ is basic and it is π -related to $[X, Y]$.
- (c) $(\nabla_{\tilde{X}}^M \tilde{Y})^h$ is basic and it is π -related to $\nabla_X^N Y$,

where ∇^N is the Levi-Civita connection of g^N .

Let us now consider the geometry of the fibers. First, we observe that the fibers are totally geodesic submanifolds of M exactly when $\mathcal{S}^\mathcal{F} = 0$. The *mean curvature* vector of the fiber is the horizontal vector field H defined by

$$(2.1) \quad H(q) = \sum_{i=1}^k \mathcal{S}^\mathcal{F}(q)(e_i, e_i) = \sum_{i=1}^k (\nabla_{e_i}^M e_i)^h$$

where $(e_i)_{i=1}^k$ is a local orthonormal frame for the fiber through q . Observe that H is not basic in general. For instance, when $n = 1$, i.e., when the fibers are hypersurfaces of M , then H is basic if and only if all the fibers have constant mean curvature. The fibers are *minimal* submanifolds of M when $H \equiv 0$.

2.2. Differential operators. Let $\pi : M \rightarrow N$ be a Riemannian submersion. Besides the natural operations of lifting a vector or vector fields in N to horizontal vectors and basic vector fields one has that functions on N can be lifted to functions on M that are constant along the fibers. Such operations preserves the regularity of the lifted objects. One can also (locally) lift curves in the base $\gamma : [a, b] \rightarrow N$ to horizontal curves $\tilde{\gamma} : [a, c] \rightarrow M$ with the same regularity as γ with arbitrary initial condition on the fiber $\mathcal{F}_{\gamma(a)}$. We will need formulas relating the derivatives of π -related objects in M and N . Let us start with divergence of vector fields.

Lemma 2.1. *Let $\tilde{X} \in \mathfrak{X}(M)$ be a basic vector field, π -related to $X \in \mathfrak{X}(N)$. The following relation holds between the divergence of \tilde{X} and X at $p \in N$ and $q \in \mathcal{F}_p$.*

$$\begin{aligned} \operatorname{div}^M(\tilde{X})_q &= \operatorname{div}^N(X)_p + g^M(\tilde{X}_q, H_q) \\ (2.2) \qquad &= \operatorname{div}^N(X)_p + g^N(d\pi_q(\tilde{X}_q), d\pi_q(H_q)). \end{aligned}$$

In particular, if the fibers are minimal, then $\operatorname{div}^M(\tilde{X}) = \operatorname{div}^N(X)$.

Proof. Formula (2.2) is obtained by a direct computation of the left-hand side, using a local orthonormal frame $e_1, \dots, e_k, e_{k+1}, \dots, e_{k+n}$ of TM , where e_1, \dots, e_k are basic fields. The equality follows using equalities (a) and (c) in Subsection 2.1, and formula (2.1) for the mean curvature. \square

Given a smooth function $f : N \rightarrow \mathbb{R}$, denote by $\tilde{f} = f \circ \pi : M \rightarrow \mathbb{R}$ its *lifting* to M . It is easy to see that the gradient $\operatorname{grad}^M \tilde{f}$ of \tilde{f} is the horizontal lifting of the gradient $\operatorname{grad}^N f$. If we denote with a tilde \tilde{X} the horizontal lifting of a vector field $X \in \mathfrak{X}(N)$, then the previous statement can be written as

$$(2.3) \qquad \operatorname{grad}^M \tilde{f} = \widetilde{\operatorname{grad}^N f}.$$

Now, given a function $f : M \rightarrow \mathbb{R}$, one can define a function $f_{\text{av}} : N \rightarrow \mathbb{R}$ by averaging f on each fiber

$$f_{\text{av}}(p) = \int_{\mathcal{F}_p} f \, d\mathcal{F}_p,$$

where $d\mathcal{F}_p$ is the volume element of the fiber \mathcal{F}_p relative to the induced metric. We are assuming that this integral is finite. As to the gradient of the averaged function f_{av} , we have the following lemma.

Lemma 2.2. *Let $p \in N$ and $v \in T_p N$ and denote by V the smooth normal vector field along \mathcal{F}_p defined by the property $d\pi_q(V_q) = v$ for all $q \in \mathcal{F}_p$. Then, for any smooth function $f : M \rightarrow \mathbb{R}$*

$$(2.4) \quad g^N(\operatorname{grad}^N f_{\text{av}}(p), v) = \int_{\mathcal{F}_p} [g^M(\operatorname{grad}^M f, V) + f \cdot g^M(H, V)] \, d\mathcal{F}_q.$$

Proof. A standard calculation as in the first variation formula for the volume functional of the fibers. Notice that when $f \equiv 1$, then f_{av} is the volume function of the fibers, and (2.4) reproduces the first variation formula for the volume. \square

Observe that, in (2.4), the gradient $\text{grad}^M f$ need not be basic or even horizontal². An averaging procedure is available also to produce vector fields X_{av} on the basis out of vector fields X defined in the total space. If $X \in \mathfrak{X}(M)$, let $X_{\text{av}} \in \mathfrak{X}(N)$ be defined by

$$(X_{\text{av}})_p = \int_{\mathcal{F}_p} d\pi_q(X_q) d\mathcal{F}_p(q).$$

Observe that the integrand above is a function on \mathcal{F}_p taking values in the fixed vector space $T_p N$. If $X \in \mathfrak{X}(M)$ is a basic vector field, π -related to the vector field $X_* \in \mathfrak{X}(N)$, then $(X_{\text{av}})_p = \text{vol}(\mathcal{F}_p) \cdot (X_*)_p$, where vol denotes the volume. Using the notion of averaged field, equality (2.4) can be rewritten as

$$\text{grad}^N(f_{\text{av}}) = (\text{grad}^M f + f \cdot H)_{\text{av}}.$$

Remark 2.3. From the above formula it follows easily that the averaged mean curvature vector field H_{av} vanishes at the point $p \in N$ if and only if p is a critical point of the function $z \mapsto \text{vol}(\mathcal{F}_z)$ in N . This happens, in particular, when the leaf \mathcal{F}_p is minimal. When all the fibers are minimal, or more generally when the averaged mean curvature vector field H_{av} vanishes identically, then the volume of the fibers is constant.

Corollary 2.4. *Let $\pi: M \rightarrow N$ be a Riemannian submersion with compact minimal fibers \mathcal{F} . Let $h \in L^2(N)$. If $f \in C_0^\infty(M)$ such that $f_{\text{av}} = 0$ for all $q \in N$ then*

$$(2.5) \quad \int_M \tilde{h} \triangle^M f dM = 0.$$

Proof. Suppose first that h is smooth. By the Divergence Theorem, Fubini's Theorem for Riemannian submersions and 2.4 we have

$$\begin{aligned} \int_M \tilde{h} \triangle^M f dM &= - \int_M g^M(\text{grad}^M \tilde{h}, \text{grad}^M f) dM \\ &= - \int_N \int_{\mathcal{F}_q} g^M(\text{grad}^M \tilde{h}, \text{grad}^M f) d\mathcal{F}_q dN \\ &= - \int_N g^N(\text{grad}^M \tilde{h}, \text{grad}^N f_{\text{av}}) dN \\ &= 0 \end{aligned}$$

²In fact, a gradient is basic if and only if it is horizontal.

If $h \in L^2(N)$ there exists a sequence of smooth functions $h_k \in C^\infty(N)$ converging to h with respect to the L^2 -norm. On the other hand

$$\begin{aligned}
\left| \int_M \tilde{h} \Delta^M f \, dM \right| &= \left| \int_M (\tilde{h}_k - \tilde{h}) \Delta^M f \, dM \right| \\
&\leq \int_M |\tilde{h}_k - \tilde{h}| |\Delta^M f| \, dM \\
&\leq \left(\int_M |\tilde{h}_k - \tilde{h}|^2 \, dM \right)^{1/2} \cdot \left(\int_M |\Delta^M f|^2 \, dM \right)^{1/2} \\
&= \|\Delta^M f\|_{L^2(M)} \cdot \left(\int_N \int_{\mathcal{F}_q} |h_k - h|^2 \, d\mathcal{F}_q \, dM \right)^{1/2} \\
&= \text{vol}(\mathcal{F}_q)^{1/2} \cdot \|\Delta^M f\|_{L^2(M)} \cdot \|h_k - h\|_{L^2(N)}
\end{aligned}$$

Since $h_k \rightarrow h$ in $L^2(N)$ then 2.5 holds. Observe that we used that the volume of the minimal fibers is constant, see Remark 2.3. \square

3. SPECTRAL ESTIMATES IN RIEMANNIAN SUBMERSIONS

3.1. Generalities on the Laplace-Beltrami operator. Let $\Omega \subset M$ be an open set in complete Riemannian manifold. The fundamental tone of Ω is defined by

$$\lambda^*(\Omega) = \inf \frac{\int_\Omega |\text{grad} f|^2}{\int_\Omega f^2},$$

where the infimum is taken over all smooth non zero functions $f \in C_0^\infty(\Omega)$.

The fundamental tone has the following monotonicity property: if $A \subset B$ are open subsets then $\lambda^*(A) \geq \lambda^*(B)$. If Ω is an open subset of M with compact closure, or if $\lambda^*(\Omega) \notin \sigma_{\text{ess}}(M)$, then $\lambda^*(\Omega)$ coincides with the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet problem

$$\begin{cases} \Delta^M u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

When $\Omega = M$ then $\lambda^*(M) = \inf \sigma(M)$.

The Laplace-Beltrami operator in N of a smooth function $f : N \rightarrow \mathbb{R}$ and the Laplace-Beltrami operator in M of its extension $\tilde{f} = f \circ \pi$ are related by the following formula.

Lemma 3.1. *Let $f : N \rightarrow \mathbb{R}$ be a smooth function and set $\tilde{f} = f \circ \pi$. Then, for all $p \in N$ and all $q \in \mathcal{F}_p$:*

$$\begin{aligned}
(\Delta^M \tilde{f})_q &= (\Delta^N f)_p + g^M((\text{grad}^M \tilde{f})_q, H_q) \\
(3.1) \quad &= (\Delta^N f)_p + g^N((\text{grad}^N f)_p, d\pi_q(H_q)).
\end{aligned}$$

The proof follows easily from (2.2) applied to the vector fields $X = \text{grad}^M \tilde{f}$ and $X_* = \text{grad}^N f$, using (2.3).

3.2. Decomposition Principle. Let $K \subset M$ be a compact set of the same dimension of M . The Laplace-Beltrami operator Δ of M acting on the space $C_0^\infty(M \setminus K)$ of smooth compactly supported functions of $M \setminus K$ has a self-adjoint extension, denoted by Δ' . The *Decomposition Principle* [11] says that $\sigma_{\text{ess}}(M) = \sigma_{\text{ess}}(M \setminus K)$. On the other hand,

$$0 \leq \lambda^*(M \setminus K) = \inf \sigma(M \setminus K) \leq \inf \sigma_{\text{ess}}(M \setminus K) = \sigma_{\text{ess}}(M),$$

thus $\mu = \sup\{\lambda^*(M \setminus K), K \subset M \text{ compact}\} \leq \sigma_{\text{ess}}(M)$. We will show that $\inf \sigma_{\text{ess}}(M) \leq \mu$. To that we will suppose that $\mu < \infty$, otherwise there is nothing to prove. Let $K_1 \subset K_2 \subset \dots$ be a sequence of compact sets with $M = \bigcup_{i=1}^\infty K_i$. We have that

$$\lambda^*(M) \leq \lambda^*(M \setminus K_1) \leq \lambda^*(M \setminus K_2) \leq \dots \rightarrow \mu.$$

Given $\varepsilon > 0$, there exists $f_1 \in C_0^\infty(M \setminus K_1)$ with $\|f_1\|_{L^2} = 1$ and $\int_M |\text{grad} f_1|^2 \leq \lambda^*(M \setminus K_1) + \varepsilon < \mu + \varepsilon$. This is $\langle (-\Delta - \mu - \varepsilon)f_1, f_1 \rangle_{L^2} < 0$. We can suppose that $\text{supp}(f_1) \subset (K_2 \setminus K_1)$. There exists $f_2 \in C_0^\infty(M \setminus K_2)$ with $\|f_2\|_{L^2} = 1$ and $\int_M |\text{grad} f_2|^2 \leq \lambda^*(M \setminus K_2) + \varepsilon < \mu + \varepsilon$. This is equivalent to $\langle (-\Delta - \mu - \varepsilon)f_2, f_2 \rangle_{L^2} < 0$. Moreover, $\int_M f_1 f_2 = 0$ since $\text{supp}(f_1) \cap \text{supp}(f_2) = \emptyset$. This way, we obtain an orthonormal sequence $\{f_k\} \subset C_0^\infty(M)$ such that $\langle (-\Delta - \mu - \varepsilon)f_k, f_k \rangle_{L^2} < 0$. By 3.3 we have that $(-\infty, \mu] \cap \sigma_{\text{ess}}(M) \neq \emptyset$ and $\inf \sigma_{\text{ess}}(M) \leq \mu$. This proves the following proposition.

Proposition 3.2. *The infimum of the essential spectrum is characterized by*

$$(3.2) \quad \inf \sigma_{\text{ess}}(M) = \sup \{ \lambda^*(M \setminus K) : K \text{ compact subset of } M \}.$$

In particular, $\sigma_{\text{ess}}(M)$ is empty if and only if given any compact exhaustion $K_1 \subset K_2 \subset \dots \subset K_n \subset \dots$ of M , the limit $\lim_{n \rightarrow \infty} \lambda^(M \setminus K_n)$ is infinite.*

Let H be a Hilbert space and $A: \mathcal{D} \subset H \rightarrow H$ be a densely defined self-adjoint operator. Given $\lambda \in \mathbb{R}$, we write $A \geq \lambda$ if $\langle Ax, x \rangle \geq \lambda \|x\|^2$ for all $x \in \mathcal{D}$. By the Spectral Theorem for (unbounded) self-adjoint operators, we have that $A \geq \lambda$ iff $\sigma(A) \subset [\lambda, +\infty)$. Let us write $A > -\infty$ if there exists $\lambda_* \in \mathbb{R}$ such that $A \geq \lambda_*$.

Lemma 3.3. *Let $A: \mathcal{D} \subset H \rightarrow H$ be a self-adjoint operator with $A > -\infty$, and let $\lambda \in \mathbb{R}$ be fixed. Assume that for all $\varepsilon > 0$ there exists an infinite dimensional subspace $G_\varepsilon \subset \mathcal{D}$ such that $\langle Ax, x \rangle < (\lambda + \varepsilon)\|x\|^2$ for all $x \in G_\varepsilon$. Then,*

$$\sigma_{\text{ess}}(A) \cap (-\infty, \lambda] \neq \emptyset.$$

This lemma is well known, see [8] but for sake of completeness we present here its proof.

Proof. First we will show that $\sigma(A) \cap (-\infty, \lambda] = \sigma(A) \cap [\lambda_*, \lambda] \neq \emptyset$. Take $\varepsilon_k = 1/k$, $k \geq 1$. By our hypothesis there exists $x_k \neq 0$ such that $\langle Ax_k, x_k \rangle < (\lambda + 1/k)\|x_k\|^2$, and thus $\sigma(A) \cap [\lambda_*, \lambda + 1/k] \neq \emptyset$ for all $k \geq 1$. Since $\sigma(A)$ is closed, it follows $\sigma(A) \cap (-\infty, \lambda] \neq \emptyset$. We may

suppose that $\sigma(A) \cap (-\infty, \lambda] \not\subset \sigma_{\text{ess}}(A)$, otherwise there is nothing to prove. Thus

$$(\sigma(A) \setminus \sigma_{\text{ess}}(A)) \cap (-\infty, \lambda] = \{\lambda_1, \dots, \lambda_n\}$$

is a finite set of eigenvalues of A of finite multiplicity. Denote by $H_i \subset \mathcal{D}$ the λ_i -eigenspace of A , $i = 1, \dots, n$, and set $X = \bigoplus_i H_i \subset \mathcal{D}$. This is clearly an invariant subspace of A . Since X has finite dimension, then $\mathcal{D} = X \oplus X_1$ where $X_1 = X^\perp \cap \mathcal{D}$ is also invariant by A . Denote by A_1 the restriction of A to the Hilbert space X_1 which is still self-adjoint. Clearly, $\sigma(A_1) = \sigma(A) \setminus \{\lambda_1, \dots, \lambda_n\}$ and $\sigma_{\text{ess}}(A_1) = \sigma_{\text{ess}}(A)$. In particular, we have $\sigma(A_1) \cap (-\infty, \lambda] \subset \sigma_{\text{ess}}(A_1)$. Using the infinite dimensionality of the space G_ε , it is now easy to see that the assumptions of our lemma hold for the operator A_1 , and the first part of the proof applies to obtain $\sigma_{\text{ess}}(A) \cap (-\infty, \lambda] = \sigma_{\text{ess}}(A_1) \cap (-\infty, \lambda] \neq \emptyset$. \square

Let us recall from [2] and [3] the following estimates for the fundamental tone of open sets of Riemannian manifolds.

Proposition 3.4. *Let $\Omega \subset M$ be an open set of a Riemannian manifold. Then*

$$(3.3) \quad \lambda^*(\Omega) \geq \frac{1}{4} \sup_X \left[\frac{\inf_\Omega \operatorname{div}(X)}{\sup_\Omega \|X\|} \right]^2,$$

where the supremum is taken over all smooth vector fields X in Ω satisfying

$$\inf_\Omega \operatorname{div}(X) > 0, \quad \text{and} \quad \sup_\Omega \|X\| < +\infty.$$

Proposition 3.5. *Let $\Omega \subset M$ be an open set of a Riemannian manifold. Given any smooth vector field $X \in \mathfrak{X}(\Omega)$ then*

$$(3.4) \quad \lambda^*(\Omega) \geq \inf_\Omega \left[\operatorname{div}(X) - |X|^2 \right].$$

Equality in (3.4) holds when $\lambda^*(\Omega) \in \sigma(\Omega) \setminus \sigma_{\text{ess}}(\Omega)$, by considering the field $X = -\nabla(\log f)$, where f is the positive eigenfunction associated to $\lambda^*(\Omega)$.

Remark 3.6. Propositions (3.4) and (3.5) hold for vector fields X of class $C^1(\Omega \setminus F) \cap L^\infty(\Omega)$ such that $\operatorname{div}(X) \in L^1(\Omega)$, where $F \subset M$ is a closed subset with $(n-1)$ -Hausdorff measure $\mathcal{H}^{n-1}(F \cap \Omega) = 0$, see [3, Lemma 3.1].

3.3. Fundamental tones estimates of Riemannian submersions. Let M and N be connected Riemannian manifolds and $\pi: M \rightarrow N$ be a Riemannian submersion. Denote by Δ^M and Δ^N the Laplacian operator on functions of (M, g^M) and of (N, g^N) respectively. We want to compare the fundamental tones of open subsets $\Omega \subset N$ with the fundamental tones of its lifting $\tilde{\Omega} = \pi^{-1}(\Omega)$.

Lemma 3.7. *Assume that the fibers of $\pi : M \rightarrow N$ are compact. Let Ω be an open subset of N , and denote by $\tilde{\Omega}$ the open subset of M given by the inverse image $\pi^{-1}(\Omega)$. Then*

$$(3.5) \quad \left[\inf_{p \in \Omega} \text{vol}(\mathcal{F}_p) \right] \cdot \lambda^*(\tilde{\Omega}) \leq \left[\sup_{p \in \Omega} \text{vol}(\mathcal{F}_p) \right] \cdot \lambda^*(\Omega).$$

In particular, if the fibers are minimal, then

$$(3.6) \quad \lambda^*(\tilde{\Omega}) \leq \lambda^*(\Omega).$$

Moreover, if $\inf_{p \in \Omega} \text{vol}(\mathcal{F}_p) > 0$ and $\sup_{p \in \Omega} \text{vol}(\mathcal{F}_p) < \infty$ then

$$(3.7) \quad \inf_{p \in \Omega} \text{vol}(\mathcal{F}_p) \cdot \inf \sigma_{\text{ess}}(M) \leq \sup_{p \in \Omega} \text{vol}(\mathcal{F}_p) \cdot \inf \sigma_{\text{ess}}(N).$$

Proof. Let $\varepsilon > 0$ and choose $f_\varepsilon \in C_0^\infty(\Omega)$ such that

$$(3.8) \quad \int_{\Omega} |\text{grad}^N f_\varepsilon|^2 < (\lambda^*(\Omega) + \varepsilon) \int_{\Omega} f_\varepsilon^2.$$

Let us consider the function $\tilde{f}_\varepsilon = f_\varepsilon \circ \pi$. By the assumption that the fibers of π are compact, \tilde{f}_ε has compact support in M . Using Fubini's Theorem for submersions we have

$$\int_{\tilde{\Omega}} |\tilde{f}_\varepsilon|^2 \, dM = \int_{\Omega} \left(\int_{\mathcal{F}_p} |\tilde{f}_\varepsilon|^2 \, d\mathcal{F}_p \right) dN = \int_{\Omega} \text{vol}(\mathcal{F}_p) \cdot |f_\varepsilon|^2 \, dN.$$

Thus

$$(3.9) \quad \int_{\tilde{\Omega}} |\tilde{f}_\varepsilon|^2 \, dM \geq \inf_{p \in \Omega} \text{vol}(\mathcal{F}_p) \cdot \int_{\Omega} |f_\varepsilon|^2 \, dN.$$

Similarly, using (2.3), we have

$$(3.10) \quad \begin{aligned} \int_{\tilde{\Omega}} |\text{grad}^M \tilde{f}_\varepsilon|^2 &= \int_{\tilde{\Omega}} |\widetilde{\text{grad}^N f_\varepsilon}|^2 \\ &= \int_{\Omega} \left(\int_{\mathcal{F}_p} |\widetilde{\text{grad}^N f_\varepsilon}|^2 \, d\mathcal{F}_p \right) dN \\ &= \int_{\Omega} \text{vol}(\mathcal{F}_p) \cdot |\text{grad}^N f_\varepsilon|^2, \end{aligned}$$

thus

$$(3.11) \quad \int_{\tilde{\Omega}} |\text{grad}^M \tilde{f}_\varepsilon|^2 \leq \sup_{p \in \Omega} \text{vol}(\mathcal{F}_p) \cdot \int_{\Omega} |\text{grad}^N f_\varepsilon|^2.$$

Using (3.8), (3.9) and (3.11), we then obtain

$$\begin{aligned}
 \inf_{p \in \Omega} \text{vol}(\mathcal{F}_p) \cdot \lambda^*(\tilde{\Omega}) &\leq \inf_{p \in \Omega} \text{vol}(\mathcal{F}_p) \cdot \frac{\int_{\tilde{\Omega}} |\text{grad}^M \tilde{f}_\varepsilon|^2}{\int_{\tilde{\Omega}} |\tilde{f}_\varepsilon|^2} \\
 (3.12) \qquad &\leq \sup_{p \in \Omega} \text{vol}(\mathcal{F}_p) \cdot \frac{\int_{\Omega} |\text{grad}^N f_\varepsilon|^2}{\int_{\Omega} |f_\varepsilon|^2} \\
 &< \sup_{p \in \Omega} \text{vol}(\mathcal{F}_p) \cdot [\lambda^*(\Omega) + \varepsilon].
 \end{aligned}$$

This proves (3.5). If all the fibers are minimal (or more generally if the averaged mean curvature vector field H_{av} vanishes identically on N , see Remark 2.3), then the volume of the fibers is constant, and inequality (3.6) follows from (3.5). To prove the inequality (3.7) we pick a compact subset $K \subset M$ and set $K_0 = \pi(K)$ and let $\tilde{K} = \pi^{-1}(K_0)$. The set \tilde{K} is compact by the assumption that the fibers of π are compact. Let $\Omega = N \setminus K_0$ and $\tilde{\Omega} = \pi^{-1}(\Omega) = M \setminus \tilde{K}$. Clearly, $\tilde{\Omega} \subset M \setminus K$ and thus $\lambda^*(\tilde{\Omega}) \geq \lambda^*(M \setminus K)$. Hence, using (3.5) we get

$$\lambda^*(M \setminus K) \leq \lambda^*(\tilde{\Omega}) \leq \frac{\sup_{p \in \Omega} \text{vol}(\mathcal{F}_p)}{\inf_{p \in \Omega} \text{vol}(\mathcal{F}_p)} \lambda^*(\Omega) \leq \frac{\sup_{p \in \Omega} \text{vol}(\mathcal{F}_p)}{\inf_{p \in \Omega} \text{vol}(\mathcal{F}_p)} \inf \sigma_{\text{ess}}(N).$$

Taking the supremum over all compact subset $K \subset M$ in the left-hand side, we obtain the desired inequality. \square

Now consider the case that the fibers of the submersion $\pi: M \rightarrow N$ are compact and minimal.

Lemma 3.8. *Let $\pi: M \rightarrow N$ be a Riemannian submersion with compact and minimal fibers \mathcal{F} . Then for every open subset $\Omega \subset N$, denoting by $\tilde{\Omega}$ the inverse image $\pi^{-1}(\Omega)$, one has that*

$$(3.13) \qquad \lambda^*(\tilde{\Omega}) = \lambda^*(\Omega).$$

Proof. In view of (3.6), it suffices to show the inequality $\lambda^*(\tilde{\Omega}) \geq \lambda^*(\Omega)$. To this aim, we will use the estimate in (3.4). We observe initially that it suffices to prove the inequality when Ω is bounded. Namely, the general case follows from $\lambda^*(\Omega) = \lim_{n \rightarrow \infty} \lambda^*(\Omega_n)$, by considering an exhaustion of Ω by a sequence of bounded open subsets Ω_n . Note that Ω is bounded if and only if $\tilde{\Omega}$ is bounded, by the compactness of the fibers. Let f be the first eigenfunction of the problem $\Delta^N u + \lambda u = 0$ in Ω with Dirichlet boundary conditions, that can be assumed to be positive in Ω .

Set $X = -\text{grad}^N(\log f)$, so that $\text{div}^N(X) - |X|^2 = \lambda_1(\Omega)$ is constant in Ω . If \tilde{X} is the horizontal lifting of X , then clearly $|\tilde{X}_q| = |X_{\pi(q)}|$ for all $q \in \tilde{\Omega}$. Moreover, by Lemma 2.1, since $H = 0$, $\text{div}^M(\tilde{X})_q = \text{div}^N(X)_{\pi(q)}$.

Using (3.4), we then obtain:

$$\lambda^*(\tilde{\Omega}) \geq \inf_{\tilde{\Omega}} [\operatorname{div}^M(\tilde{X}) - |\tilde{X}|^2] = \inf_{\Omega} [\operatorname{div}^N(X) - |X|^2] = \lambda^*(\Omega).$$

This proves Lemma (3.8). \square

Corollary 3.9. *Assume that the fibers of π are compact and minimal. Then, $\sigma_{\text{ess}}(M) = \emptyset$ if and only if $\sigma_{\text{ess}}(N) = \emptyset$.*

The above result applies in particular to Riemannian coverings, yielding the following

Corollary 3.10. *If M is a finite covering of N , then $\sigma_{\text{ess}}(M) \neq \emptyset$ if and only if $\sigma_{\text{ess}}(N) \neq \emptyset$.*

3.4. Proof of Theorem 1. The item ii. of Theorem 1 follows from this Lemma 3.8. For if we take a sequence of compact sets $K_1 \subset K_2 \subset \dots$ with $N = \bigcup_{i=1}^{\infty} K_i$. Likewise we have $M = \bigcup_{i=1}^{\infty} \tilde{K}_i$, where $\tilde{K}_i = \pi^{-1}(K_i)$. By the proof of (3.2) we have that $\inf \sigma_{\text{ess}}(N) = \lim_{i \rightarrow \infty} \lambda^*(N \setminus K_i)$ and $\inf \sigma_{\text{ess}}(M) = \lim_{i \rightarrow \infty} \lambda^*(M \setminus \tilde{K}_i)$. However, $\lambda^*(N \setminus K_i) = \lambda^*(M \setminus \tilde{K}_i)$, by the Lemma (3.8). Before we prove item i. we need the following lemma.

Lemma 3.11. *Let $\pi: M \rightarrow N$ be a Riemannian submersion with compact minimal fibers \mathcal{F} . If $f \in L^2(N)$ and $\Delta^N f \in L^2(N)$ then $\tilde{f} \in L^2(M)$ and $\Delta^M \tilde{f} = \widetilde{\Delta^N f} \in L^2(M)$. In other words, if $f \in \operatorname{Dom}(\Delta^N)$ then $\tilde{f} \in \operatorname{Dom}(\Delta^M)$.*

Proof. Let $\tilde{f} = f \circ \pi$ be the lifting of f . By Fubini's Theorem we have

$$\int_M \tilde{f}^2 dM = \int_N \left(\int_{\mathcal{F}_p} f^2 d\mathcal{F}_p \right) dN = \operatorname{vol}(\mathcal{F}_p) \int_N f^2 dN < \infty.$$

This shows that $\tilde{f} \in L^2(M)$. To show that $\Delta^M \tilde{f} \in L^2(M)$ we will show that $\Delta^M \tilde{f} = \widetilde{\Delta^N f}$. Every $\varphi \in C_0^\infty(M)$ can be decomposed as $\varphi = \varphi_1 + \varphi_2$ where φ_1 is constant along the fibers \mathcal{F} and $(\varphi_2)_{\text{av}} = 0$, see [5]. Moreover, φ_1 and φ_2 has compact support. Observe that we can define $\psi: N \rightarrow \mathbb{R}$ by $\psi(\pi(p)) = \varphi_1(p)$ so that $\varphi_1 = \tilde{\psi}$. By the Lemma 3.1 we have that $\Delta^M \varphi_1(p) = \Delta^N \psi(\pi(p))$ for every $p \in M$. By Corollary 2.4

$\int_M \tilde{f} \Delta^M \varphi_2 dM = 0$, therefore

$$\begin{aligned}
\int_M \tilde{f} \Delta^M \varphi dM &= \int_M \tilde{f} \Delta^M \varphi_1 dM \\
&= \int_N \left(\int_{\mathcal{F}_p} f \Delta^M \varphi_1 d\mathcal{F}_p \right) dN \\
&= \int_N \left(f \Delta^N \psi \int_{\mathcal{F}_p} d\mathcal{F}_p \right) dN \\
&= \text{vol}(\mathcal{F}_p) \int_N f \Delta^N \psi dN \\
&= \text{vol}(\mathcal{F}_p) \int_N \psi \Delta^N f dN \\
&= \int_N \left(\int_{\mathcal{F}_p} \psi \Delta^N f d\mathcal{F}_p \right) dN \\
&= \int_M \widetilde{\psi \Delta^N f} dM \\
&= \int_M \varphi_1 \widetilde{\Delta^N f} dM \\
&= \int_M \varphi \widetilde{\Delta^N f} dM
\end{aligned}$$

□

To show that $\sigma_p(N) \subset \sigma_p(M)$ we take $\lambda \in \sigma_p(N)$ and $f \in L^2(N)$ with $-\Delta^N f = \lambda f$ in distributional sense. This implies that $\widetilde{-\Delta^N f} = \lambda \tilde{f}$. By Lemma 3.11, $-\Delta^M \tilde{f} = \lambda \tilde{f}$ showing that $\lambda \in \sigma_p(M)$. To show that $\sigma_{\text{ess}}(N) \subset \sigma_{\text{ess}}(M)$ we take $\mu \in \sigma_{\text{ess}}(N)$. There exists an orthonormal sequence of functions $f_k \in \text{Dom}(\Delta)$ such that $\| -\Delta^N f_k - \mu f_k \|_{L^2(N)} \rightarrow 0$ as $k \rightarrow \infty$. By Lemma 3.11, we have that $\tilde{f}_k \in \text{Dom}(\Delta^M)$. Now

$$\begin{aligned}
\| -\Delta^M \tilde{f}_k - \mu \tilde{f}_k \|_{L^2(M)}^2 &= \int_M | -\Delta^M \tilde{f}_k - \mu \tilde{f}_k |^2 dM \\
&= \int_N \int_{\mathcal{F}_q} | -\Delta^N f_k - \mu f_k |^2 d\mathcal{F}_q dN \\
&= \text{vol}(\mathcal{F}_q) \int_N | -\Delta^N f_k - \mu f_k |^2 dN \\
&= \text{vol}(\mathcal{F}_q) \| -\Delta^N f_k - \mu f_k \|_{L^2(N)}^2 \rightarrow 0
\end{aligned}$$

This shows that $\mu \in \sigma_{\text{ess}}(M)$, the proof of Theorem 1 is concluded.

Corollary 3.12. *Let G be a Lie group endowed with a bi-invariant metric. Then, $\sigma_{\text{ess}}(G)$ is empty if and only if for some (hence for any) compact subgroup $K \subset G$, the Riemannian homogeneous space G/K has empty essential spectrum.*

Proof. Apply Theorem 1 to the Riemannian submersion $G \mapsto G/K$, which has minimal and compact fibers. \square

Other interesting examples of applications of Theorem 1 arise from non compact Lie groups.

Example. Consider the 2×2 special linear group $\mathrm{SL}(2, \mathbb{R})$. There exists a 2-parameter family of left-invariant Riemannian metrics $g_{\kappa, \tau}$, with $\kappa < 0$ and $\tau \neq 0$, for which $(\mathrm{SL}(2, \mathbb{R}), g_{\kappa, \tau}) \rightarrow \mathbb{H}^2(\kappa)$ is a Riemannian submersion with geodesic fibers diffeomorphic to the circle \mathbb{S}^1 . An explicit description of these metrics can be found, for instance, in [24]. Endowed with these metrics, $\mathrm{SL}(2, \mathbb{R})$ is one of the eight homogeneous Riemannian 3-geometries, as classified in [22], and its isometry group has dimension 4.

Proposition 3.13. For all $\kappa < 0$ and $\tau \neq 0$,

$$\sigma(\mathrm{SL}(2, \mathbb{R}), g_{\kappa, \tau}) = \sigma_{\mathrm{ess}}(\mathrm{SL}(2, \mathbb{R}), g_{\kappa, \tau}) = \left[-\frac{\kappa}{4}, +\infty\right).$$

Proof. It is known that the spectrum $\sigma(\mathbb{H}(\kappa)) = \sigma_{\mathrm{ess}}(\mathbb{H}(\kappa)) = \left[-\frac{\kappa}{4}, +\infty\right)$, see [8]. By Lemma 3.8

$$\lambda^*(\mathrm{SL}(2, \mathbb{R}), g_{\kappa, \tau}) = \lambda^*(\mathbb{H}(\kappa)) = -\frac{\kappa}{4},$$

hence $\sigma(\mathrm{SL}(2, \mathbb{R}), g_{\kappa, \tau}) \subset \left[-\frac{\kappa}{4}, +\infty\right)$. On the other hand, by Theorem 1

$$\left[-\frac{\kappa}{4}, +\infty\right) = \sigma_{\mathrm{ess}}(\mathbb{H}(\kappa)) \subset \sigma_{\mathrm{ess}}(\mathrm{SL}(2, \mathbb{R}), g_{\kappa, \tau}).$$

This proves the proposition. \square

4. MEAN CURVATURE OF GEODESIC SPHERES VERSUS MEAN CURVATURE OF THE FIBERS. PROOF OF THEOREM 2.

We will now drop the minimality and the compactness assumption on the fibers, however, we will make some assumptions on the curvature of the base and the fibers of the submersion. Assume that (N, g^N) has a *pole* p_0 or more generally has a point p_0 with *thin cut locus*, see the Introduction. For $p \in N \setminus \{p_0\}$, let $\gamma_p: [0, 1] \rightarrow N$ be the unique affinely parameterized geodesic in (N, g^N) such that $\gamma_p(0) = p_0$ and $\gamma_p(1) = p$. The *radial curvature* function of (N, g^N, p_0) , denoted by $\kappa_{p_0}: N \rightarrow \mathbb{R}$, is defined by $\kappa_{p_0}(p) = \max_{\sigma} \sec(\sigma)$, where \sec is the section curvature and the maximum is taken over all 2-planes $\sigma \subset T_p N$ containing the direction $\gamma_p'(1)$. Finally, let us denote by $\rho_{p_0}: N \rightarrow [0, +\infty)$ the distance function in N given by $\rho_{p_0}(p) = \mathrm{dist}_N(p, p_0)$.

We are now ready for

Proof of Theorem 2. Assume first that $\pi: M \rightarrow N$ is a Riemannian submersion satisfying the following assumptions:

- (a) (N, g^N) has a pole p_0 .
- (b) the function $h(q) = (\Delta^N \rho_{p_0})_{\pi(q)} + g^N((\mathrm{grad}^N \rho_{p_0})_{\pi(q)}, d\pi_q(H_q))$ is proper.

Consider the lifting $\tilde{\rho}_{p_0} : M \rightarrow \mathbb{R}$ defined by $\tilde{\rho}_{p_0} = \rho_{p_0} \circ \pi$. Then by Lemma 3.1 we have that $h = \triangle^M \tilde{\rho}_{p_0}$. Moreover, by (2.3),

$$|\text{grad}^M \tilde{\rho}_{p_0}| = |\text{grad}^N \rho_{p_0}| \equiv 1.$$

If $K_1 \subset K_2 \subset \dots \subset K_n \subset \dots$ is a compact exhaustion of M , then by (3.3) applied to $X = \text{grad}^M \tilde{\rho}_{p_0}$

$$(4.1) \quad \lambda^*(M \setminus K_n) \geq \frac{1}{4} \left[\inf_{M \setminus K_n} h \right]^2.$$

Since h is proper, the right-hand side in the above inequality tends to $+\infty$ as $n \rightarrow \infty$, thus, by Proposition 3.2, $\sigma_{\text{ess}}(M) = \emptyset$.

If p_0 has thin cut locus, the same proof above holds, since $X = \text{grad}^M \tilde{\rho}_{p_0}$ satisfies the Proposition 3.4 and therefore 4.1, see Remark 3.6. \square

Corollary 4.1. *If the fibers of $\pi : M \rightarrow N$ are compact and if the function $l(p) = \triangle^N \rho_{p_0}(p) - \max_{q \in \mathcal{F}_p} \|H_q\|$ is proper then $\sigma_{\text{ess}}(M) = \emptyset$.*

A different statement can be obtained in terms of radial curvature.

Corollary 4.2. *Assume that $G : [0, +\infty[\rightarrow \mathbb{R}$ is a smooth function such that:*

$$(4.2) \quad \kappa_{p_0}(p) \leq -G(\rho_{p_0}(p))$$

for all $p \in N$. Denote by $\psi : [0, +\infty[\rightarrow \mathbb{R}$ the solution of the Cauchy problem:

$$\psi''(t) = G(t)\psi(t), \quad \psi(0) = 0, \quad \psi'(0) = 1,$$

and set $\ell(t) = (n-1)\psi'(t)/\psi(t)$, $t > 0$. If

$$(4.3) \quad \lim_{q \rightarrow \infty} \left[\ell(\rho_{p_0}(\pi(q))) + g^N((\nabla^N \rho_{p_0})_{\pi(q)}, d\pi_q(H_q)) \right] = +\infty$$

then $\sigma_{\text{ess}}(M) = \emptyset$.

Proof. Using the Hessian Comparison Theorem [14, Chapter 2], under the assumption (4.2) one has:

$$(4.4) \quad \text{Hess}(\rho^N) \geq \frac{\psi'}{\psi} \cdot (g^N - d\rho_{p_0} \otimes d\rho_{p_0}).$$

Considering an orthogonal basis of $T_p N$ of the form $\{\nabla^N \rho_{p_0}, e_1, \dots, e_{n-1}\}$, where $\{e_1, \dots, e_{n-1}\}$ is an orthonormal basis of $T_p \mathcal{S}_p$, and taking the trace of the symmetric bilinear forms in the two sides of (4.4), we get

$$(4.5) \quad \triangle^N \rho_{p_0} \geq (n-1) \frac{\psi'}{\psi}.$$

It is clear that (4.3) implies that $h(q) = \triangle^N \rho_{p_0} + g^N(\text{grad}^N \rho_{p_0}, d\pi_q(H_q))$ is proper. \square

4.1. Warped products. Let (N, g^N) and (F, g^F) be Riemannian manifolds and let $\psi : N \rightarrow \mathbb{R}^+$ be a smooth function. The *warped product manifold* $M = N \times_\psi F$ is the product manifold $N \times F$ endowed with the Riemannian metric $g^N + \psi^2 g^F$. It is immediate to see that the projection $\pi : M \rightarrow N$ onto the first factor is a Riemannian submersion, with fiber $\mathcal{F}_p = \{p\} \times F$. Among Riemannian submersions, warped products are characterized by the following properties:

- the horizontal distribution is integrable, and its leaves are totally geodesic;
- the fibers are totally umbilical.

For warped products, the results of the paper can be stated in a more explicit form in terms of the warping function f . The mean curvature of the fibers are given by

$$(4.6) \quad H = -\dim(F) \frac{\text{grad}^M \tilde{\psi}}{\tilde{\psi}},$$

where $\tilde{\psi}$ is the lifting of ψ .

Proposition 4.3. *Let $M = N \times_\psi F$ be a warped product, with F compact.*

- If $\sigma_{\text{ess}}(N) \neq \emptyset$, and $0 < \inf_N \psi \leq \sup_N \psi < +\infty$.
Then $\sigma_{\text{ess}}(M) \neq \emptyset$.*
- If (N, g^N) has a pole p_0 and the function*

$$\Delta^N \rho_{p_0} - \frac{1}{\psi} g^N(\nabla^N \rho_{p_0}, \nabla^N \psi)$$

is proper, then $\sigma_{\text{ess}}(M) \neq \emptyset$.

Proof. Part (a) follows from Proposition 3.7, observing that the volume of the fiber $\mathcal{F}_p = \{p\} \times F$ equals $\psi(p)^{\dim(F)} \text{vol}(F)$.

Part (b) follows from Theorem 2 and formula (4.6). \square

4.2. Example. Let $\mathbb{R}^n = [0, \infty) \times \mathbb{S}^{n-1} / \sim$ endowed with the smooth metric $ds^2 = dt^2 + f^2(t) d\theta^2$, where $f(0) = 0$, $f'(0) = 1$. The equivalence relation \sim is the following

$$(t, \theta) \sim (s, \alpha) \Leftrightarrow t = s = 0 \text{ or } t = s > 0 \text{ and } \theta = \alpha.$$

The radial sectional curvatures K^{rad} along the geodesic issuing the origin $0 = \{0\} \times \mathbb{S}^{n-1} / \sim$ is given by $K^{\text{rad}}(t, \theta) = -\frac{f''(t)}{f(t)}$. Let us consider

$W = \mathbb{R}^n \times \mathbb{S}^1$ with metric $dw^2 = ds^2 + g^2(\rho(x)) d_{\mathbb{S}^1}^2$, where ρ is the distance function to the origin in (\mathbb{R}^n, ds^2) and $g : [0, \infty) \rightarrow (0, \infty)$ is a smooth function. Choosing $f(t) = te^{t^2}$ we have $K^{\text{rad}}(t, \theta) = -4t^3 - 6t$, thus $\lim_{t \rightarrow \infty} K^{\text{rad}}(t, \theta) = -\infty$. By Donnelly-Li's Theorem [11], the spectrum of (\mathbb{R}^n, ds^2) is discrete. Choosing $g(t) = e^{t-t^2}$. An easy computation yields that

- The volume $\text{vol}(W, dw^2) = \infty$.

- ii. The limit $\mu = \limsup_{r \rightarrow \infty} \frac{\log(\text{vol}(B_W(r)))}{r} < \infty$, where $B_W(r)$ is the geodesic ball centered at a point $p = (0, \xi) \in W$ and radius r .

The items i. and ii. imply by Brooks' Theorem [6] that $\sigma_{\text{ess}}(W) \neq \emptyset$. This gives an example of a Riemannian submersion $\pi: (W, dw^2) \rightarrow (\mathbb{R}^n, ds^2)$ where the base space is discrete but the total space is not, while the fiber is compact but not minimal.

An example of a Riemannian submersion $\pi: (\mathbb{R}^n \times \mathbb{S}^1, dw^2) \rightarrow (\mathbb{R}^n, ds^2)$ where the total space is discrete but the base space is not, while the fiber is compact but not minimal is presented in [1, Proposition 4.3].

REFERENCES

- [1] A. BAIDER, *Noncompact Riemannian manifolds with discrete spectra*, J. Diff. Geom. **14** (1979), 41–57.
- [2] G. P. BESSA, J. F. MONTENEGRO, *Eigenvalue estimates for submanifolds with locally bounded mean curvature*, Ann. Global Anal. Geom. **24** (2003), 279–290.
- [3] G. P. BESSA, J. F. MONTENEGRO, *An extension of Barta's theorem and geometric applications*, Ann. Global Anal. Geom. **31** (2007), no. 4, 345–362.
- [4] G. P. BESSA, L. JORGE, J. F. MONTENEGRO, *The spectrum of the Martin-Morales-Nadirashvili minimal surfaces is discrete*. to appear in J. Geom. Anal.
- [5] M. BORDONI, *Spectral estimates for submersions with fibers of basic mean curvature*, An. Univ. Vest. Timiș. Ser. Mat.-Inform. **44** (2006), no. 1, 23–36.
- [6] R. BROOKS, *A relation between growth and the spectrum of the Laplacian*, Math. Z. **178** (1981), 501–508.
- [7] E. B. DAVIES, *Spectral theory and differential operators*, Cambridge University Press, 1995.
- [8] H. DONNELLY, *On the essential spectrum of a complete Riemannian manifold*, Topology **20** (1981), no. 1, 1–14.
- [9] H. DONNELLY, *Negative curvature and embedded eigenvalues*. Math. Z. **203**, (1990), 301–308.
- [10] H. DONNELLY, N. GAROFALO, *Riemannian manifolds whose Laplacian have purely continuous spectrum*. Math. Ann. **293**, (1992), 143–161.
- [11] H. DONNELLY, P. LI, *Pure point spectrum and negative curvature for noncompact manifolds*, Duke Math. J. **46** (1979), 497–503.
- [12] E. ESCOBAR, *On the spectrum of the Laplacian on complete Riemannian manifolds*. Comm. Partial Differ. Equations **11**, (1985), 63–85.
- [13] A. GRAY, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. **16** (1967), 715–737.
- [14] R. E. GREENE, H. WU, *Function theory on manifolds which possess a pole*, Lecture Notes in Mathematics, 699. Springer, Berlin, 1979.
- [15] M. HARMER, *Discreteness of the spectrum of the Laplaceian and Stochastic incompleteness*, J. Geom. Anal. **19** (2009), 358–372.
- [16] L. KARP, *Noncompact manifolds with purely continuous spectrum*, Mich. Math. J. **31**, (1984), 339–347.
- [17] R. KLEINE, *Discreteness conditions for the Laplacian on complete noncompact Riemannian manifolds*, Math. Z. **198** (1988), 127–141.
- [18] R. KLEINE, *Warped products with discrete spectra*, Results Math. **15** (1989), 81–103.
- [19] B. O'NEILL, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.
- [20] ———, *Submersions and geodesics*, Duke Math. J. **34** (1967), 363–373.

- [21] F. Rellich, *Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten*. Jahresber. Dtsch. Math.-Ver. **53**, (1943), 57–65.
- [22] P. SCOTT, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (5) (1983), 401–487.
- [23] T. TAYOSHI, *On the spectrum of the Laplace-Beltrami operator on noncompact surface*, Proc. Japan Acad. **47**, (1971), 579–585.
- [24] F. TORRALBO, *Rotationally invariant constant mean curvature surfaces in homogeneous 3-manifolds*, preprint 2009, arXiv:0911.5128v1.

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